

A MULTIPLICITY BOUND FOR GRADED RINGS AND A CRITERION FOR THE COHEN-MACAULAY PROPERTY

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ABSTRACT. Let R be a polynomial ring over a field. We prove an upper bound for the multiplicity of R/I when I is a homogeneous ideal of the form $I = J + (F)$, where J is a Cohen-Macaulay ideal and $F \notin J$. The bound is given in terms of two invariants of R/J and the degree of F . We show that ideals achieving this upper bound have high depth, and provide a purely numerical criterion for the Cohen-Macaulay property. Applications to quasi-Gorenstein rings and almost complete intersections are given.

1. INTRODUCTION

Bounds on the multiplicity of a ring S in terms of other invariants of S (and analyses of the rings achieving these bounds) have attracted a strong interest over the last 130 years, from the classical lower bound $\deg X \geq \text{codim } X + 1$ for non-degenerate projective varieties to the still open Eisenbud-Green-Harris Conjecture [3], or the Huneke-Srinivasan Multiplicity Conjecture, proved a few years ago by Eisenbud and Schreyer [5].

In the present paper, we prove a new upper bound for the multiplicity of a wide class of graded rings and study the defining ideals achieving this bound. Let R be a polynomial ring over a field k , J a homogeneous Cohen-Macaulay ideal, $F \notin J$ a homogeneous element and $I = J + (F)$. Let $e(S)$ denote the multiplicity of a graded ring S . If F is regular on R/J , it is well-known that $e(R/I) = e(R/J) \cdot \deg(F)$. If F is a zero-divisor on R/J (that is, $\text{ht } I = \text{ht } J$) one has the elementary inequality $e(R/I) \leq e(R/J) - 1$. In the present paper, we prove the following sharper upper bound

$$(1.1) \quad e(R/I) \leq e(R/J) - \max\{1, s - \deg(F) + 1\},$$

where $s = s(R/J)$ is the difference between the smallest graded shift appearing in the last step of a minimal graded free resolution of R/J and $\text{ht } J$. This bound, by its nature, is more restrictive when $\deg(F)$ is small or when R/J is level, a particular instance of which is when R/J is Gorenstein.

The inequality (1.1) generalizes a previous result of Engheta bounding the multiplicity of a homogeneous almost complete intersection in terms of the degrees of its minimal generators [6, Theorem 1], see Corollary 2.3.

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We name ideals I achieving equality in (1.1) *ideals of maximal multiplicity*. Our second main result states that these ideals define factor rings with high depth. More precisely, R/I is Cohen-Macaulay if the degree of F is sufficiently small, namely, $\deg(F) \leq s$. In the remaining cases, that is, when $\deg(F) > s$, we show $\text{depth}(R/I) = \dim(R/I) - 1$, provided R/J is Gorenstein (Theorem 2.7). A consequence of this result is that almost complete intersections generated in a single degree and having maximal multiplicity are Cohen-Macaulay (Corollary 2.9), a result that we employ in [10]. Also, one should contrast this result with the abundance of examples of almost complete intersection ideals I (even generated in a single degree) of any other multiplicity that are not Cohen-Macaulay (see Remark 2.10).

We then provide a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity, and exhibit examples, which include rational normal curves and \mathfrak{m} -primary ideals (see Theorem 2.13, Examples 2.15 and 2.17 and Corollary 2.16). Among the applications, we use linkage to deduce a lower bound for the multiplicity of any graded quasi-Gorenstein ring S in terms of its a -invariant and dimension (Proposition 3.2). If this lower bound is achieved, then S is actually Gorenstein (see Theorem 3.3). We also prove that ideals of maximal multiplicity are unmixed if and only if they are Cohen-Macaulay (Corollary 3.5).

As a final application, we remark that part of the material from this paper is employed in the forthcoming paper [10] where, motivated by a question of Stillman [18], we prove a close-to-optimal upper bound on the projective dimension of any ideal generated by 4 quadratic polynomials. One of our original motivations in the present paper was, in fact, to find a more structural reason for the fact (proved in an earlier draft of [10]) that any almost complete intersection ideal (not necessarily unmixed) of multiplicity 6 generated by 4 quadrics is Cohen-Macaulay.

The structure of the paper is the following: in Section 2, we prove the three main results, namely the upper bound (1.1), the high depth properties of ideals of maximal multiplicity and a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. We employ these results to obtain a sufficient condition for almost complete intersections of quadrics to be Cohen-Macaulay and provide examples. In Section 3, we prove a lower bound for the multiplicity of quasi-Gorenstein rings, a multiplicity-based sufficient condition for quasi-Gorenstein rings to be Gorenstein, and analogies between ideals of maximal multiplicity and ideals of multiplicity one.

2. THE MAIN RESULTS

Throughout this paper R is a polynomial ring over a field k and \mathfrak{m} denotes its unique homogeneous maximal ideal. Since we may harmlessly replace k by $k(X)$, by base change we may always assume $|k| = \infty$.

Recall that the *Hilbert function* of a finitely generated graded R -module M is the numerical function given by $HF_M(i) = \dim_k M_i$ for all $i \in \mathbb{Z}$. If A is a graded artinian factor ring of R , then $e(A) = \sum_{i=0}^{\infty} HF_A(i)$, and the *socle* of A is the k -vector space $\text{Soc}(A) = 0 :_A \mathfrak{m}_A$, where $\mathfrak{m}_A = \mathfrak{m}A$. A *socle element* is an element of $\text{Soc}(A)$.

Let J be a homogeneous Cohen-Macaulay R -ideal of height g . We consider the invariant

$$s(R/J) = \min\{i \mid \text{Tor}_g^R(R/J, k)_i \neq 0\} - g.$$

We remark here two additional interpretations of this invariant. First, when $|k| = \infty$, $s(R/J)$ is the smallest degree of a non-zero homogeneous socle element of a general artinian reduction of R/J (a *general artinian reduction* of $T = R/J$ is a ring $T/(L_1, \dots, L_d)$ where L_i are general linear forms and $d = \dim(T)$). Second, one has $s(R/J) = c(R/J) + \dim(R/J)$, where $c(R/J) = -\max\{i \mid [k \otimes_{R/J} \omega_{R/J}]_i \neq 0\}$ and $\omega_{R/J}$ denotes the graded canonical module of R/J .

We also recall that the *unmixed part* of an R -ideal K , denoted K^{un} , is the intersection of the primary components of K of minimal height. The ideal K is *unmixed* if $K = K^{un}$. The associativity formula yields $e(R/K) = e(R/K^{un})$ and the following easy remark:

Remark 2.1. If $K \subseteq L$ are unmixed ideals of the same height, then $e(R/K) \geq e(R/L)$, and $e(R/K) = e(R/L)$ if and only if $K = L$.

We now prove our first main result.

Theorem 2.2. *Let J be a homogeneous Cohen-Macaulay R -ideal, and let $F \notin J$ be homogeneous. Set $I = J + (F)$ and assume $\text{ht } I = \text{ht } J$. Then*

- (i) $e(R/I) \leq e(R/J) - \max\{1, s(R/J) - \deg(F) + 1\}$;
- (ii) *if equality is achieved in (i) and $\deg(F) \leq s(R/J)$, then R/I is Cohen-Macaulay.*

Proof. We prove both statements at once. Set $\delta = \deg(F)$ and $s = s(R/J)$. First, assume $1 > s - \delta + 1$. We need to show $e(R/I) \leq e(R/J) - 1$. This follows by the equality $e(R/I) = e(R/I^{un})$, the inclusions $J \subsetneq I \subseteq I^{un}$ and Remark 2.1. Hence, we may assume $1 \leq s - \delta + 1$, that is, $\delta \leq s$. We need to show that $e(R/I) \leq e(R/J) - (s - \delta + 1)$ and, if equality is achieved, then R/I is Cohen-Macaulay.

Let A be a general artinian reduction of R/J , and denote by $\bar{}$ images in A . We claim that both statements hold if $\bar{F} \neq 0$ in A . Indeed, in this case, we set $n = \max\{t \in \mathbb{N}_0 \mid \bar{F} \mathfrak{m}_A^t \neq 0\} < \infty$ and observe that $HF_{A/(\bar{F})}(i) \leq HF_A(i) - 1$ for every $\delta \leq i \leq \delta + n$. This proves $e(A/(\bar{F})) \leq e(A) - (n + 1)$. Let G be a homogeneous element of degree n such that $\bar{F}G \neq 0$ in A . By definition of n , we have $0 \neq \bar{F}G \in \text{Soc}(A)$, whence $\delta + n = \deg(\bar{F}G) \geq s$. This proves $s - \delta \leq n$ and gives $e(A/(\bar{F})) \leq e(A) - (n + 1) \leq e(A) - (s - \delta + 1)$.

A well-known result of Serre [1, Theorem 4.7.10] yields $e(A) = e(R/J)$ and $e(R/I) \leq e(A/(\overline{F}))$, hence we obtain the inequalities

$$e(R/I) \leq e(A/(\overline{F})) \leq e(A) - (s - \delta + 1) = e(R/J) - (s - \delta + 1).$$

If, moreover, $e(R/I) = e(R/J) - (s - \delta + 1)$, then we have the equality $e(R/I) = e(A/(\overline{F}))$ and, by [1, Theorem 4.7.10], R/I is Cohen-Macaulay.

Therefore, to finish the proof it suffices to prove $\overline{F} \neq 0$ in A . Set $\tilde{R} = R/J$ and let \sim denote images in $\tilde{R} = R/J$. Since A is a general artinian reduction of \tilde{R} , to show $\overline{F} \neq 0$ in A , we need to prove that \tilde{F} is not in the intersection of all the (general) minimal reductions of $\tilde{\mathfrak{m}}$, that is, $\tilde{F} \notin \text{core}(\tilde{\mathfrak{m}})$ in $\tilde{R} = R/J$, where $\text{core}(\mathfrak{m})$ denotes the core of \mathfrak{m} . Since $\tilde{R} = R/J$ is Cohen-Macaulay, it follows by work of Fouli, Polini and Ulrich, [8, Corollary 4.3.(b)] that $\text{core}(\tilde{\mathfrak{m}}) \subseteq \tilde{\mathfrak{m}}^{s+1}$. The inequality $s \geq \delta$ now yields $\tilde{F} \notin \text{core}(\tilde{\mathfrak{m}})$ for degree reasons. \square

As a first application we recover an upper bound (first proved by Engheta [6]) for the multiplicity of R/I , where I is any homogeneous almost complete intersection.

Corollary 2.3. (Engheta, [6, Theorem 1]) *Let $I = (f_1, \dots, f_{g+1})$ be a homogeneous almost complete intersection of height g in R , where f_1, \dots, f_g form a homogeneous regular sequence. Set $d_i = \deg(f_i)$ for every $i = 1, \dots, g+1$. Then,*

$$e(R/I) \leq \prod_{i=1}^g d_i - \max \left\{ 1, \sum_{i=1}^g (d_i - 1) - (d_{g+1} - 1) \right\}.$$

In particular, if I is generated by forms of the same degree d and $g > 1$ then

$$e(R/I) \leq d^g - (d - 1)(g - 1).$$

Proof. Since $J = (f_1, \dots, f_g)$ is a complete intersection, we have $s(R/J) = \sum_{i=1}^g (d_i - 1)$. Now, Theorem 2.2 applied to $I = J + (f_{g+1})$ proves the statement. \square

Although the bound of Corollary 2.3 can be sharp (see Remark 2.10) G. Caviglia remarked that, if this is the case, then either $g = 2$, or $g = 3$ and $d_1 = d_2 = d_3 = 2$. Instead, the bound of Theorem 2.2 is more general and, indeed, is achieved in a wider variety of situations (see, for instance, Examples 2.15 and 2.17 and Corollary 2.16).

Before stating our second main result, we recall a few definitions. An ideal I is called *almost Cohen-Macaulay* if $\text{depth}(R/I) \geq \dim(R/I) - 1$. Also, a homogeneous Cohen-Macaulay ideal J of height g is said to be *level* if there exists only one positive integer i such that $[\text{Tor}_g^R(R/J, k)]_i \neq 0$. An ideal J is *Gorenstein* if R/J is Gorenstein, that is, $\text{Tor}_g^R(R/J, k) \cong k$. Clearly, every homogeneous Gorenstein ideal is level.

We will also need a few definitions and results from liaison theory. Two (homogeneous) ideals J and K in R are *linked*, denoted $J \sim K$, if there

exists a (homogeneous) Gorenstein ideal G with $K = G :_R J$ and $J = G :_R K$. Sometimes we say J and K are linked *by* G . Linkage (by complete intersections) has been studied since the nineteenth century, although its first modern treatment appeared in the ground-breaking paper by Peskine and Szpiro [19]. Properties of liaison by Gorenstein ideals were then studied in [23]. We refer the interested reader to [16], [12] and their references.

Proposition 2.4. (Peskine-Szpiro, Schenzel [19], [23]) *Let J be an unmixed ideal of R of height g . If $G \subseteq J$ is a height g Gorenstein ideal, and $K = G : J$, then $J \sim K$.*

We will use the following simple fact: If G is a Gorenstein ideal contained in an ideal J and $\text{ht } G = \text{ht } J$, then $G : J = G : J^{un}$, that is, $G : J$ is linked to J^{un} .

Proposition 2.5. (Peskine-Szpiro, Golod, Schenzel [19], [9], [23]) *If $J \sim K$, then*

- (a) *S/J is Cohen-Macaulay if and only if S/K is Cohen-Macaulay;*
- (b) *$e(S/J) + e(S/K) = e(S/G)$, where G is the ideal defining the link $J \sim K$.*

The following linkage result is well-known (its proof follows, for instance, along the same lines of the proof of [7, Theorem 3]).

Lemma 2.6. *Let $I = J + (F)$, where J is Gorenstein ideal with $\text{ht}(J) = \text{ht}(I)$, and $F \notin J$. If L is any ideal linked to I^{un} , then $\text{pd}(R/I) \leq \text{pd}(R/L) + 1$.*

In the setting of Theorem 2.2, we say that I has *maximal multiplicity* if there exist a Cohen-Macaulay ideal J and an element $F \notin J$ with $I = J + (F)$, $\text{ht}(I) = \text{ht}(J)$ and $e(R/I) = e(R/J) - \max\{1, s(R/J) - \deg(F) + 1\}$. If J and F are as above, we say they form a *maximal decomposition* of I .

We can now state our second main result, proving the high depth of R/I .

Theorem 2.7. *Let I be a homogeneous R -ideal of maximal multiplicity, and let $I = J + (F)$ be a maximal decomposition of I .*

- (a) *If $\deg(F) \leq s(R/J)$, then R/I is Cohen-Macaulay.*
- (b) *If R/J is level, then R/I is Cohen-Macaulay if and only if $\deg(F) \leq s(R/J)$.*
- (c) *If $\deg(F) > s(R/J)$ and R/J is Gorenstein, then $\text{depth}(R/I) = \dim(R/I) - 1$.*

Note that, although J is Cohen-Macaulay, the ideal I may not even be unmixed.

Proof. Set $g = \text{ht } J = \text{ht } I$. Assertion (a) was proved in Theorem 2.2 (ii). To prove assertion (b) we need to show that if R/I is Cohen-Macaulay, then $\deg(F) \leq s(R/J)$. Let L_1, \dots, L_d be general linear forms, where $d = \dim(R/J) = \dim(R/I)$, and let $-$ denote images modulo L_1, \dots, L_d . Since

R/I is Cohen-Macaulay, by Theorem [1, Theorem 4.7.10], we have $e(R/I) = e(\overline{R}/\overline{I})$. Moreover, since R/J is level, we have $\text{Soc}(\overline{R}/\overline{J}) = \overline{\mathfrak{m}}_{R/J}^s$, where $s = s(R/J)$. Now, assume by contradiction $\deg(F) > s(R/J)$. Then $\overline{F} \in \overline{\mathfrak{m}}_{R/J}^{s+1} = 0$ in $\overline{R}/\overline{J}$, that is, $\overline{R}/\overline{I} = \overline{R}/\overline{J}$. This implies

$$e(R/I) = e(\overline{R}/\overline{I}) = e(\overline{R}/\overline{J}) = e(R/J).$$

Since both I and J are unmixed, Remark 2.1 implies $J = I$, which is a contradiction.

We now prove assertion (c). The assumptions imply $e(R/I) = e(R/J) - 1$. Let $L = J : I$ and note that, by Lemma 2.6, $\text{pd}(R/I) \leq \text{pd}(R/L) + 1$. Moreover, R/L is unmixed with $e(R/L) = e(R/J) - e(R/I) = 1$ (by Proposition 2.5), therefore, by a well-known result of Samuel (see Proposition 3.4), R/L is Cohen-Macaulay. Then $\text{pd}(R/I) \leq g + 1$. Hence, by the Auslander-Buchsbaum formula, we have $\text{depth}(R/I) \geq \dim(R/I) - 1$. Finally, by assertion (b), R/I is not Cohen-Macaulay, because $\deg(F) > s(R/J)$. This yields $\text{depth}(R/I) = \dim(R/I) - 1$. \square

The next simple example shows that the bound given in Theorem 2.2 can be (trivially) sharp, and there are ideals of maximal multiplicity that are not Cohen-Macaulay.

Example 2.8. Let $R = k[x, y, z]$, $J = (x^2, xy, y^2)$ and $F = xz$. Then $I = J + (F)$ is not unmixed, R/I has maximal multiplicity and is almost Cohen-Macaulay.

We now apply Theorem 2.7 to almost complete intersection ideals generated by quadrics.

Corollary 2.9. Let I be an almost complete intersection of height $g \geq 1$ generated by homogeneous elements of the same degree 2. If $e(R/I) = 2^g - (g - 1)$, then R/I has maximal multiplicity and is Cohen-Macaulay.

Proof. It follows by Corollary 2.3 together with Theorem 2.7 (b). \square

If I is an almost complete intersection generated by 4 quadrics (that is, $g = 3$ and $d = 2$), then Corollary 2.3 gives $e(R/I) \leq 6$, and Corollary 2.9 yields that R/I is Cohen-Macaulay if $e(R/I) = 6$. This fact is employed in [10].

The following remark shows that, without further assumptions, no other values of $e(R/I)$ guarantee the Cohen-Macaulayness of R/I .

Remark 2.10. Let I be a height three ideal generated by four quadrics.

- (a) If $e(R/I) = 6$, then R/I is Cohen-Macaulay;
- (b) For every $1 \leq e \leq 6$, there are examples of I with $e(R/I) = e$ and, if $e \neq 6$, R/I is not Cohen-Macaulay.

Assertion (a) follows by Corollary 2.9. Assertion (b) can be seen, for instance, as follows. Take $R = k[a, b, c, x, y, z]$,

- (1) if $I = (ax, by, cz, x^2 + y^2 + z^2)$, then $e(R/I) = 1$ and $\text{pd}(R/I) = 4$;

- (2) if $I = (ax, by, xy + xz + yz, x^2 + y^2 + z^2)$, then $e(R/I) = 2$ and $\text{pd}(R/I) = 4$;
- (3) if $I = (ax + by + cz, x^2, y^2, z^2)$, then $e(R/I) = 3$ and $\text{pd}(R/I) = 6$;
- (4) if $I = (ax, x^2, y^2, z^2)$, then $e(R/I) = 4$ and $\text{pd}(R/I) = 4$;
- (5) if $I = (ax + by + cz, bx + cy + az, cx + ay + bz, bx + cy - bz - cz)$, then $e(R/I) = 5$ and $\text{pd}(R/I) = 4$;
- (6) if $I = (x^2, y^2, z^2, xy)$, then $e(R/I) = 6$ and, by part (a), $\text{pd}(R/I) = \text{ht } I = 3$.

Next, we want to provide a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. The first step consists in describing the structure of the colon ideal $J : F$.

Lemma 2.11. *Let I be a Cohen-Macaulay homogeneous ideal of height g having maximal multiplicity. If J and F form a maximal decomposition of I , then $J : F = (x_1, \dots, x_{g-1}, q)$, for some linearly independent linear forms x_1, \dots, x_{g-1} and an element $q \notin (x_1, \dots, x_{g-1})$.*

Proof. From the short exact sequence

$$(\star) \quad 0 \longrightarrow R/J : F[-\deg(F)] \longrightarrow R/J \longrightarrow R/I \longrightarrow 0$$

one obtains $e(R/J) - e(R/I) = e((R/J : F)[- \deg(F)]) = e(R/J : F)$. First, assume I is \mathfrak{m} -primary. If $\deg(F) \geq s(R/J)$, then, by assumption of maximal multiplicity, $e(R/I) = e(R/J) - 1$, whence $e(R/J : F) = 1$. Then $J : F = \mathfrak{m}$ and the statement follows. We may then assume $\deg(F) < s(R/J)$. Since J and F form a maximal decomposition of I , the proof of Theorem 2.2 gives $HF_{R/I}(i) = HF_{R/J}(i) - 1$ for all $\deg(F) \leq i \leq s(R/J)$. This yields that $HF_{R/J:F}(i) = 1$ if $0 \leq i \leq s(R/J) - \deg(F)$, hence, there exist linearly independent linear forms x_1, \dots, x_{g-1}, x_g such that $J : F = (x_1, \dots, x_{g-1}, x_g^c)$ and the statement follows.

Now assume $\dim(R/I) = d > 0$. Let L_1, \dots, L_d be general linear forms in R , set $L = (L_1, \dots, L_d)$ and let $\bar{}$ denote images in $\bar{R} = R/L$. Since I is Cohen-Macaulay, L_1, \dots, L_d form a regular sequence on R/I , then, applying the functor $-\otimes_R R/L$ to the short exact sequence (\star) , one obtains the short exact sequence

$$0 \longrightarrow \bar{R}/\bar{J} : \bar{F} \longrightarrow \bar{R}/\bar{J} \longrightarrow \bar{R}/\bar{I} \longrightarrow 0.$$

Clearly, $\bar{R} = R/L$ is still a polynomial ring, \bar{I} is a homogeneous $\bar{\mathfrak{m}}$ -primary ideal, \bar{R}/\bar{I} has maximal multiplicity, and \bar{J} and \bar{F} form a maximal decomposition of \bar{I} . Then, by the above, one has $\bar{J} : \bar{F} = \bar{J} : \bar{F} = (\bar{z}_1, \dots, \bar{z}_{g-1}, \bar{z}_g^c)$ for some linearly independent linear forms z_1, \dots, z_g of \bar{R} . The statement now follows by lifting this equality back to R . \square

For the rest of the paper, a complete intersection C of height g containing $g - 1$ linearly independent linear forms is called an *almost linear complete intersection*.

Proposition 2.12. *Let I be a Cohen-Macaulay homogeneous ideal of height g having maximal multiplicity. Then there exists an almost linear complete intersection C' such that $(I + C')/C'$ is cyclic.*

Proof. If I is contained in an almost linear complete intersection, then $(I + C')/C' = 0$ and the statement follows trivially. We may then assume I is not contained in any almost linear complete intersection. Let J and F form a maximal decomposition of I . By Lemma 2.11, the ideal $C' = J : F$ is an almost linear complete intersection. Note that $J \subseteq I \cap C'$. Since I is not contained in C' , then $(I + C')/C'$ is non-zero. Now, the natural mapping $I/J \rightarrow I/I \cap C' \cong (I + C')/C' \rightarrow 0$ together with the assumption that I/J is cyclic yields that also $(I + C')/C'$ is cyclic. \square

Next, we prove a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. The assumption on $(I + C')/C'$ being cyclic is necessary, by Lemma 2.12.

Theorem 2.13. *Let I be a homogeneous Cohen-Macaulay ideal of height g . If there exists an almost linear complete intersection C' satisfying the following two conditions:*

- (i) $(I + C')/C'$ is non-zero and cyclic, generated by an element of degree $\delta \geq 1$, and
- (ii) $e(R/C') \leq \max\{1, s(R/I) - \delta + 1\}$,

then I has maximal multiplicity.

Proof. Let F be a homogeneous element of I of degree $\delta \geq 1$ whose image generates the cyclic module $(I + C')/C'$. Set $J = C' \cap I$ and note that $I = J + (F)$. Set $C = J : F = C' : F$. Since $C' \subseteq C$ are unmixed of the same height, the ideal C is again an almost linear complete intersection and $e(R/C) \leq e(R/C')$. Write $C = (x_1, \dots, x_{g-1}, q)$ and note that $e(R/C) = \deg(q) \leq e(R/C') \leq \max\{1, s(R/I) - \delta + 1\}$. We need to show that J is Cohen-Macaulay and $e(R/I) = e(R/J) - \max\{1, s(R/I) - \delta + 1\}$.

Since I and C are Cohen-Macaulay ideals of height g , the short exact sequence

$$(\star) \quad 0 \longrightarrow (R/C)[- \delta] \xrightarrow{\cdot F} R/J \longrightarrow R/I \longrightarrow 0$$

implies that R/J is Cohen-Macaulay too. Now, if $1 \geq s(R/I) - \delta + 1$, then $e(R/C) \leq 1$ and, since C is a proper ideal, then $e(R/C) = 1$, yielding that $e(R/I) = e(R/J) - e(R/J : F) = e(R/J) - 1$. This proves that I has maximal multiplicity. We may then assume $s(R/I) - \delta + 1 > 1$. The Horseshoe Lemma applied to (\star) gives $s(R/J) \geq \min\{s(R/I), s(R/C[- \delta])\}$. Since C is an almost linear complete intersection, we have

$$s(R/C[- \delta]) = \delta + s(R/C) = \delta + (\deg(q) + g - 1) - g = \delta + \deg(q) - 1,$$

whence we obtain

$$s(R/J) \geq \min\{s(R/I), s(R/C[- \delta])\} = \min\{s(R/I), \delta + \deg(q) - 1\} = \delta + \deg(q) - 1,$$

where the last equality holds because the inequalities

$$\deg(q) = e(R/C) \leq e(R/C') \leq \max\{1, s(R/I) - \delta + 1\} = s(R/I) - \delta + 1$$

imply that $\delta + \deg(q) - 1 \leq s(R/I)$. Then, we have obtained $\delta + \deg(q) - 1 \leq s(R/J)$, that is, $\deg(q) \leq s(R/J) - \delta + 1$. We now apply Theorem 2.2 and obtain

$$\begin{aligned} e(R/J) - e(R/C) &= e(R/I) && \leq e(R/J) - \max\{1, s(R/J) - \delta + 1\} \\ &\leq e(R/J) - \max\{1, \deg(q)\} && = e(R/J) - \deg(q) \\ &= e(R/J) - e(R/C') \end{aligned}$$

proving that $e(R/I) = e(R/J) - \max\{1, s(R/J) - \delta + 1\}$. \square

Note that the proof of Theorem 2.13 is constructive, in the sense that if $(I + C')/C'$ is non-zero, cyclic and $e(R/C') \leq \max\{1, s(R/I) - \delta + 1\}$, then one can explicitly construct a maximal decomposition $I = J + (F)$ of the ideal I .

We isolate the special case where C' is a *linear prime*, that is, where C' is a prime ideal generated by linear forms.

Corollary 2.14. *Let I be a homogeneous Cohen-Macaulay ideal of height g . If there exists a linear prime C' of height g such that $(I + C')/C'$ is cyclic and non-zero, then I has maximal multiplicity.*

Proof. By assumption $e(R/C') = 1$, so the multiplicity condition of Theorem 2.13 is trivially satisfied. \square

We now exhibit two classes of ideals having maximal multiplicity.

Example 2.15. *Let I be any homogeneous \mathfrak{m} -primary ideal, then I has maximal multiplicity.*

Proof. Let F be any minimal generator of I , and set $J = I + F \cdot \mathfrak{m}$. Then I has maximal multiplicity because $I = J + (F)$ and $e(R/I) = e(R/J) - 1$. \square

Also ideals generated by the 2 by 2 minors of catalecticant matrices have maximal multiplicity. For instance, ideals defining rational normal curves have maximal multiplicity.

Corollary 2.16. *Fix integers $d \geq 1$, $r \geq 2$ and $N \geq 3$. Then the ideal $I = I_2(A)$ generated by the 2-minors of the matrix*

$$A = \begin{pmatrix} x_1^d & x_2^d & \cdots & x_{N-r+2}^d \\ x_2^d & x_3^d & \cdots & x_{N-r+1}^d \\ \vdots & \vdots & \ddots & \vdots \\ x_r^d & x_{r+1}^d & \cdots & x_{N+1}^d \end{pmatrix}$$

has maximal multiplicity.

In particular, the defining ideals of the rational normal curves of \mathbb{P}^N for any $N \geq 3$ have maximal multiplicity.

Proof. It is known that $\text{ht}(I) = N - 1$. Set $C' = (x_2, \dots, x_N)$, and note that $(I + C')/C'$ is cyclic and non-zero, generated by the image of $F = x_1^d x_{N+1}^d - x_r^d x_{N-r+2}^d$ in R/C' . Then, by Corollary 2.14, I has maximal multiplicity, and a maximal decomposition of I is given by $I = J + (F)$, where J is the ideal generated by all the 2 by 2 minors of A except for F . Rational normal curves consist of the special case where $r = 2$ and $d = 1$. \square

We remark that, in contrast with the ideals I satisfying equality in Corollary 2.3, these ideals can be generated in arbitrarily high degrees and can have arbitrarily large heights.

We conclude this section with a class of almost complete intersections I of height 3, generated in a single degree $d \geq 4$ and having maximal multiplicity (compare with the discussion after Corollary 2.3). Note that, for these ideals, there is no linear prime C' of height 3 such that $(I + C')/C'$ is cyclic, hence one cannot use Corollary 2.14 to prove that I has maximal multiplicity.

Example 2.17. Fix $t \geq 1$, let f_i, g_i, h_i , where $i = 1, 2$, be irreducible polynomials of the same degree $t + 1$ in disjoint sets of variables such that f_1 is contained in a linear prime of height 2. Set $F = h_1 h_2$ and $J = (f_1 f_2, g_1 g_2, f_1 g_1 h_1)$. Then $I = J + (F)$ is a Cohen-Macaulay almost complete intersection that has maximal multiplicity.

Proof. We first show that I is Cohen-Macaulay. Observe that the ideals (f_l, g_i, h_j) with $1 \leq l \leq 2, 1 \leq i \leq 2$ and $1 \leq j \leq 2$ are all prime. Since

$$I = \left(\bigcap_{1 \leq i \leq 2, 1 \leq j \leq 2} (f_1, g_i, h_j) \right) \cap (f_2, g_1, h_1) \cap (f_2, g_1, h_2) \cap (f_2, g_2, h_1),$$

then I is unmixed. Also, since $C_0 = (f_1 f_2, g_1 g_2, h_1 h_2)$ is a complete intersection of height 3 contained in I , then $I \sim C_0 : I$ by Proposition 2.4. Since $C_0 : I = (f_2, g_2, h_2)$ is a complete intersection, then, by Proposition 2.5, the ideal I is Cohen-Macaulay.

Let (x_1, x_2) be a linear prime of height 2 containing f_1 , and set $C' = (x_1, x_2, g_1)$. Then $(I + C')/C'$ is cyclic, generated by the image of $F = h_1 h_2$ in R/C' . We have $s(R/I) = 5t + 2$, hence $s(R/I) - \deg(F) + 1 = (5t + 2) - (2t + 2) + 1 = 3t + 1 > 3$ for every $t \geq 1$. Then

$$e(R/C') = 2 < 3t + 1 = \max\{1, s(R/I) - \deg(F) + 1\}.$$

Then I has maximal multiplicity by Theorem 2.13. \square

For instance, let $R = k[x_1, \dots, x_8, y_1, \dots, y_8, z_1, \dots, z_8]$, $f_1 = x_1^t x_2 - x_3^t x_4$, $g_1 = y_1^t y_2 - y_3^t y_4$, $h_1 = z_1^t z_2 - z_3^t z_4$, $f_2 = x_5^t x_6 - x_7^t x_8$, $g_2 = y_5^t y_6 - y_7^t y_8$, $h_2 = z_5^t z_6 - z_7^t z_8$, and $F = h_1 h_2$. Then $I = (f_1 f_2, g_1 g_2, f_1 g_1 h_1, F)$ is a Cohen-Macaulay almost complete intersection that has maximal multiplicity and is generated in degree $2t + 2 \geq 4$ for any $t \geq 1$.

3. APPLICATION TO QUASI-GORENSTEIN RINGS

In this section we prove a lower bound for the multiplicity of quasi-Gorenstein rings and a sufficient condition for a quasi-Gorenstein ring to be Gorenstein.

We first recall the definition of graded quasi-Gorenstein rings (also known as 1-Gorenstein rings). Recall that the canonical module of a d -dimensional graded ring S with homogeneous maximal ideal \mathfrak{m}_S is defined as $\omega_S = \text{Hom}_k(H_{\mathfrak{m}_S}^d(S), E)$, where $E = E_S(S/\mathfrak{m}_S)$ is the injective envelope of the residue field of S .

A graded ring S is *quasi-Gorenstein* if $\omega_S \cong S(a)$ for some integer a . An ideal Q is said to be *quasi-Gorenstein* if R/Q is quasi-Gorenstein. The number $a = a(R/Q)$ is the *a-invariant* of R/Q . Note that quasi-Gorenstein ideals are unmixed (indeed, their factor rings satisfy Serre's property (S_2)).

Quasi-Gorenstein rings arise naturally in several contexts, for instance, (extended) Rees algebras (cf. [24, Theorem 2.8], [14, Theorem 3.2], and [11, Theorem 6.1]) or coordinate rings of cones over abelian surfaces (see, for instance, [21]). From the definition, it follows that an ideal Q of R is Gorenstein if and only if Q is quasi-Gorenstein and Cohen-Macaulay. We will employ the following result, essentially proved by Schenzel [22, Proposition 1].

Proposition 3.1. *Let Q be an R -ideal. The following are equivalent:*

- (i) Q is quasi-Gorenstein;
- (ii) *there exist a Gorenstein ideal $G \subseteq Q$ and $h \in R$ so that $Q \sim G + hR$ by G .*

We now give a lower bound for the multiplicity of graded quasi-Gorenstein rings.

Proposition 3.2. *If Q is a homogeneous quasi-Gorenstein ideal, then*

$$e(R/Q) \geq \max \{1, a(R/Q) + \dim(R/Q) + 1\}.$$

Proof. We may assume Q is proper. By Proposition 3.1, there exists a Gorenstein ideal $G \subseteq Q$ with $\text{ht } G = \text{ht } Q$ and $Q \sim I$, where $I = G + hR$ for some element h . Note that $h \notin G$ (otherwise $I = G$ and then, by Proposition 2.4, $Q = G : G = R$, contradicting Q is proper) and $\text{ht } I = \text{ht } G$. Then, by Proposition 2.5 (b) and Theorem 2.2, we have

$$e(R/Q) = e(R/G) - e(R/I) \geq \max\{1, s(R/G) - \deg(h) + 1\}.$$

Now, from the standard exact sequence from linkage

$$0 \rightarrow G \rightarrow I \rightarrow \omega_{R/Q}(-a(R/G)) \rightarrow 0$$

we have $\omega_{R/Q} \cong I/G[a(R/G)]$. Since I/G is generated by the image of h , we obtain $a(R/Q) = a(R/G) - \deg(h)$. This fact, together with the equalities

$s(R/G) = \dim(R/G) + a(R/G)$ and $\dim(R/G) = \dim(R/Q)$, yields

$$\begin{aligned} e(R/Q) &\geq \max\{1, s(R/G) - \deg(h) + 1\} \\ &= \max\{1, \dim(R/G) + a(R/G) - \deg(h) + 1\} \\ &= \max\{1, \dim(R/Q) + a(R/Q) + 1\}. \end{aligned}$$

□

We now prove a multiplicity-based sufficient condition for R/Q to be Gorenstein.

Theorem 3.3. *Let Q be a homogeneous quasi-Gorenstein ideal. If*

$$e(R/Q) = \max\{1, a(R/Q) + \dim(R/Q) + 1\},$$

then R/Q is Gorenstein.

Proof. We may assume Q is proper. Let $G \subseteq Q$ be a Gorenstein ideal with $\text{ht } G = \text{ht } Q$, and set $I = G : Q$. From the proof of Proposition 3.2, the given equality implies that I has maximal multiplicity.

Since $\deg(h) = a(R/G) - a(R/Q)$ and $-a(R/Q) \leq \dim(R/Q)$, then we have $\deg(h) \leq s(R/G)$. By Theorem 2.7 (b), this yields that R/I is Cohen-Macaulay. Thus, by Proposition 2.5 (a), R/Q is Cohen-Macaulay and, then, Gorenstein. □

Next, we would like to point out an analogy between the two extremal values of $e(R/I)$. Assume $I = J + (F)$, where J is Gorenstein and $F \notin J$ is a zero divisor on R/J , then

$$1 \leq e(R/I) \leq e(R/J) - \max\{1, s(R/J) - \deg(F) + 1\}.$$

If I has maximal multiplicity, by Theorem 2.7 either R/I is Cohen-Macaulay or is almost Cohen-Macaulay. When $e(R/I) = 1$ a similar statement holds.

Proposition 3.4. *Let J be a homogeneous Gorenstein ideal, and let $F \notin J$ be a homogeneous element such that $\text{ht } I = \text{ht } J$, where $I = J + (F)$. Assume $e(R/I) = 1$.*

- (a) (Samuel [20], [17]) *I is unmixed if and only if I is Cohen-Macaulay.*
- (b) *I is not unmixed if and only if $\text{depth}(R/I) = \dim(R/I) - 1$ (that is, R/I is almost Cohen-Macaulay).*

Proof. We only prove assertion (b). If $\text{depth}(R/I) = \dim(R/I) - 1$, then R/I is not Cohen-Macaulay, hence, by assertion (a), I is not unmixed. Next, assume I is not unmixed. Set $L = J : I$ and note that $L \sim I^{un}$, by Proposition 2.4. Since $e(R/I^{un}) = e(R/I) = 1$, the ideal I^{un} is Cohen-Macaulay, by assertion (a). By Proposition 2.5 (a), R/L is also Cohen-Macaulay. Then, Lemma 2.6 gives $\text{pd}(R/I) \leq \text{grade } I + 1$. An application of the Auslander-Buchsbaum formula now proves $\text{depth}(R/I) \geq \dim(R/I) - 1$. Since I is not unmixed, we have $\text{depth}(R/I) \neq \dim(R/I)$, yielding $\text{depth}(R/I) = \dim(R/I) - 1$. □

It is then natural to ask whether the analogue of Proposition 3.4 holds: assume I has maximal multiplicity, is it true that I is unmixed if and only if I is Cohen-Macaulay? Corollary 3.5 provides a positive answer.

Corollary 3.5. *Let J be a homogeneous Gorenstein ideal, and let $F \notin J$ be a homogeneous element such that $\text{ht } I = \text{ht } J$, where $I = J + (F)$. Assume I has maximal multiplicity.*

- (a) *I is unmixed if and only if I is Cohen-Macaulay if and only if $\deg(F) \leq s(R/J)$.*
- (b) *I is not unmixed if and only if $\text{depth}(R/I) = \dim(R/I) - 1$ if and only if $\deg(F) > s(R/J)$.*

Proof. (a) By Theorem 2.7 (c) we only need to prove that, if I is unmixed, then I is Cohen-Macaulay. If I is unmixed, the ideal $Q = J : I$ is quasi-Gorenstein by Proposition 3.1. As in the proof of Proposition 3.2, the maximal multiplicity of I yields $e(R/Q) = \max \{1, a(R/Q) + \dim(R/Q) + 1\}$. Now, Theorem 3.3 implies that Q is Gorenstein and then, by Proposition 2.5 (a), I is Cohen-Macaulay.

Assertion (b) follows from assertion (a) and Theorem 2.7 (a) and (c). \square

We conclude with a question. If I is an ideal of maximal multiplicity, in general there could be several different maximal decompositions $I = J + (F)$. Moreover, if F and F' are minimal generators of I , there could be a maximal decomposition of I of the form $I = J + (F)$, but no maximal decomposition for I of the form $I = J' + (F')$. These observations raise the following question: *Is there an implicit characterization of ideals I of maximal multiplicity?* Theorem 2.13 gives a sufficient condition when I is Cohen-Macaulay, and Proposition 2.12 gives a necessary condition, but we don't know a necessary and sufficient condition..

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REFERENCES

- [1] W. Bruns and J. Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, Vol. **39**. Cambridge University Press, Cambridge, 1993.
- [2] CoCoA, Computations in Commutative Algebra. Available at <http://cocoa.dima.unige.it/>
- [3] D. Eisenbud, M. Green and J. Harris, Higher Castelnuovo theory, *Astérisque* **218** (1993), 187–202.
- [4] D. Eisenbud and J. Harris, On varieties of minimal degree (a centennial account), *Proc. Sympos. Pure Math.* **46** (1987), 3–13.
- [5] D. Eisenbud and F. O. Schreyer, Betti numbers of graded modules and cohomology of vector bundles, *J. Amer. Math. Soc.* **22** (2009), 859–888.

- [6] B. Engheta, Bound on the multiplicity of almost complete intersections, *Comm. Alg.* **37** (2009) 948–953.
- [7] B. Engheta, A bound on the projective dimension of three cubics, *J. Symb. Comp.* **45** (2010) 60–73.
- [8] L. Fouli, C. Polini and B. Ulrich, Annihilator of graded components of the canonical module, and the core of standard graded algebras, *Trans. Amer. Math. Soc.* **362** (2010), 6183–6203.
- [9] E. S. Golod, A note on perfect ideals, *Algebra* (A. I. Kostrikin, ed.), Moscow State Univ. Publishing House, 1980, 37–39.
- [10] C. Huneke, P. Mantero, J. McCullough and A. Seceleanu, Ideals generated by 4 quadric polynomials, preprint.
- [11] W. Heinzer, M.-K. Kim and B. Ulrich, The Cohen-Macaulay and Gorenstein properties of rings associated to filtrations, *Comm. Alg.* **39** (2011), 3547–3580.
- [12] C. Huneke and B. Ulrich, The structure of linkage, *Ann. of Math.* **126** (1987), 277–334.
- [13] C. Huneke and B. Ulrich, General hyperplane sections of Algebraic Varieties, *J. Alg. Geom.* **2** (1993), 487–505.
- [14] M. Johnson and B. Ulrich, Serre’s condition (R_k) for associated graded rings, *Proc. Amer. Math. Soc.* **127** (1999), 2619–2624.
- [15] Macaulay 2, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2/>
- [16] J. C. Migliore, *Introduction to Liaison Theory and Deficiency Modules*, Birkhuser, *Progress in Mathematics* **165**, (1998).
- [17] M. Nagata, *Local rings*, Robert E. Kreiger Publishing Co., 1975.
- [18] I. Peeva and M. Stillman, Open problems on syzygies and Hilbert functions, *J. Commut. Algebra* **1** (2009), no. 1, 159–195.
- [19] C. Peskine and L. Szpiro, Liaison des variétés algébriques, *Invent. Math.* **26** (1974), 271–302.
- [20] P. Samuel, La notion de multiplicité en algèbre et en géométrie algébrique, I & II, *J. Math. Pures. Appl.* **30** (1951), 159–274.
- [21] P. Schenzel, On Buchsbaum rings and their canonical modules, *Seminar Eisenbud-Singh-Vogel*, Vol. 1, *Teubner-Texte Math.* **29**, Teubner, Leipzig, 1980, 65–76.
- [22] P. Schenzel, A note on almost complete intersections, *Seminar Eisenbud-Singh-Vogel*, Vol. 2, *Teubner-Texte Math.* **48**, Teubner, Leipzig, 1982, 49–54.
- [23] P. Schenzel, Notes on liaison and duality, *J. Math. Kyoto Univ.* **22** (1982), 485–498.
- [24] O. Lavila-Vidal and S. Zarzuela, On the Gorenstein property of the diagonals of the Rees algebra, *Collect. Math.* **49** (1998), 383–397.

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